Levelling-Up **Basic Mathematics**

Powers

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The aim of this document is to provide a short, self assessment programme for students who wish to acquire a basic competence in the use of powers.

Last Revision Date: January 16, 2001 Version 1.1

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1. Powers (Introduction)

If a is any number and n any *positive* integer (whole number) then the product of a with itself n times, $\underbrace{a \times a \times \cdots \times a}_{n}$, is called *$a$ raised to the power n*, and written a^n , i.e.,

$$a^n = \underbrace{a \times a \times \cdots \times a}_{n}.$$

Examples 1

(a) $7^2 = 7 \times 7 = 49$

(b) $2^3 = 2 \times 2 \times 2 = 8$

(c) $3^5 = 3 \times 3 \times 3 \times 3 \times 3 = 243$

The following important rules apply to powers.

$$\begin{array}{lrcl}
1. & a^m \times a^n & = & a^{m+n} \\
2. & a^m \div a^n & = & a^{m-n} \\
3. & (a^m)^n & = & a^{mn} \\
4. & a^1 & = & a \\
5. & a^0 & = & 1
\end{array}$$

We want these rules to be true for all positive values of a and all values of m and n. We shall first look at the simpler cases.

Examples 2

(a) $10^2 \times 10^3 = (10 \times 10) \times (10 \times 10 \times 10) = 10^5 = 10^{2+3}$.

(b) $2^5 \div 2^3 = 32 \div 8 = 4 = 2^2 = 2^{5-3}$.

(c) $(3^2)^3 = (3 \times 3)^3 = (3 \times 3) \times (3 \times 3) \times (3 \times 3) = 3^6 = 3^{2\times 3}$

(d) From rule 2 $a^{n+1} \div a^n = a^{(n+1)-n} = a^1$. Also

$$a^{n+1} \div a^n = \frac{\overbrace{a \times a \times \cdots \times a \times a}^{n+1}}{\underbrace{a \times a \times \cdots \times a}_{n}} = a = a^1\,.$$

(e) We have $a^n \times a^0 = a^{n+0} = a^n = a^n \times 1$. Thus $a^0 = 1$.

Exercise

Simplify each of the following.

1. $2^3 \times 2^3$
2. $3^{15} \div 3^{12}$
3. $(10^2)^3$

2. Negative Powers

The question now arises as to what we mean by a *negative* power. To interpret this note that

$$a^2 \div a^5 = \frac{a^2}{a^5} = \frac{a \times a}{a \times a \times a \times a \times a} = \frac{1}{a^3}\,.$$

If rule 2 is to apply, then $a^2 \div a^5 = a^{2-5} = a^{-3}$. Thus $a^{-3} = 1/a^3$. The general rule is

$$\boxed{a^{-n} = 1/a^n\,.}$$

Examples 3

(a) $10^{-2} = 1/10^2 = 1/100$

(b) $3^{-1} = 1/3^1 = 1/3$

(c) $5^2 \div 5^4 = 5^{(2-4)} = 5^{-2} = 1/5^2 = 1/25$

Exercise

Write each of the following in the form a^k, for some number k.

1. $2^3 \times 2^{-5}$
2. $3^5 \div 3^7$
3. $(10^2)^{-3}$

3. Fractional Powers

If a is a *positive* number, then the *square root* of a is the number which, multiplied by itself, gives a. Thus 3 is the square root of 9 since $3^2 = 9$. We write $3 = \sqrt{9}$. Note that, by definition, $\sqrt{a} \times \sqrt{a} = a$. This gives us a way of interpreting $a^{\frac{1}{2}}$ for, by rule 1,

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{(\frac{1}{2}+\frac{1}{2})} = a^1 = a = \sqrt{a} \times \sqrt{a}$$

so that $a^{\frac{1}{2}} = \sqrt{a}$. The general rule is that, if a is a positive number and n is a positive integer, then

$$\boxed{a^{\frac{1}{n}} = \sqrt[n]{a}}$$

where $\sqrt[n]{a}$ is the n-th root of a. We can see this in general for, by rule 3,

$$(a^{\frac{1}{n}})^n = a^{\frac{1}{n} \times n} = a^1 = a.$$

Examples 4

(a) $100^{\frac{1}{2}} = \sqrt{100} = 10$

(b) $8^{\frac{1}{3}} = \sqrt[3]{8} = 2$

(c) $27^{\frac{5}{3}} = (27^{\frac{1}{3}})^5 = 3^5 = 243$

In (c) we have used rule 3, i.e. $a^{\frac{m}{n}} = a^{\frac{1}{n}\times m} = (a^{\frac{1}{n}})^m$, so

$$(a^{\frac{1}{n}})^m = a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}}$$

Quiz. To which of the following does $(8^5)^{\frac{1}{3}}$ simplify?

(a) 8 (b) 16 (c) 24 (d) 32

4. Use of the Rules of Simplification

In this section we shall demonstrate the use of the rules of powers to simplify more complicated expressions.

Examples 5

Simplify each of the following.

1. $\left[(a^{-3})^{\frac{2}{3}}\right]^{\frac{1}{2}}$

2. $\left[\left(x^{-\frac{1}{4}}\right)^{8}\right]^{\frac{2}{3}}$

3. $\left(x^{\frac{1}{2}}\right)^{3} \times \left(x^{-\frac{1}{3}}\right)^{2}$,

4. $\left(\sqrt[4]{x^3}\right)^{\frac{2}{3}} \times \left(\sqrt[5]{x^6}\right)^{\frac{5}{12}}$

5. $\left(\frac{a^2}{b^3}\right)^{\frac{1}{3}} \times \left(\frac{b^2}{a^3}\right)^{\frac{1}{2}}$

5. Quiz on Powers

Begin Quiz

1. $\left(\sqrt[3]{a^5}\right)^{\frac{1}{2}} \times \sqrt[6]{a^{-5}}$

(a) 1 (b) a (c) $a^{\frac{5}{12}}$ (d) $a^{\frac{5}{6}}$

2. $\left(\frac{a^3}{b^2}\right)^{\frac{1}{2}} \div \left(\frac{b^3}{a^2}\right)^{-\frac{1}{2}}$

(a) $a^{-\frac{1}{2}}b^{-\frac{1}{2}}$ (b) $a^{\frac{1}{2}}b^{-\frac{1}{2}}$ (c) $a^{-\frac{1}{2}}b^{\frac{1}{2}}$ (d) $a^{\frac{1}{2}}b^{\frac{1}{2}}$

3. $\left(\sqrt[4]{b^3}\right)^{\frac{1}{6}} \times \sqrt[9]{b^{-3}} \div \left(\sqrt{b^{-7}}\right)^{\frac{1}{7}}$

(a) $b^{\frac{1}{8}}$ (b) $b^{-\frac{1}{8}}$ (c) $b^{\frac{3}{8}}$ (d) $b^{-\frac{3}{8}}$

End Quiz

Score: Correct

Solutions to Quizzes

Solution to Quiz:

Using rule 3, we have

$$(8^5)^{\frac{1}{3}} = 8^{(5\times\frac{1}{3})} = 8^{(\frac{1}{3}\times 5)} = (8^{\frac{1}{3}})^5 = 2^5 = 32.$$

End Quiz

Solutions to Problems

Problem 1. $2^3 \times 2^3 = 2^{(3+3)} = 2^6 = 64$ $\square$

Problem 2. $3^{15} \div 3^{12} = 3^{(15-12)} = 3^3 = 27$ $\square$

Problem 3. $(10^2)^3 = 10^{(2\times 3)} = 10^6 = 1,000,000$ $\square$

Problem 1. $2^3 \times 2^{-5} = 2^{(3-5)} = 2^{-2}$, which is $1/4$. $\square$

Problem 2. $3^5 \div 3^7 = 3^{(5-7)} = 3^{-2}$, which is $1/9$. $\square$

Problem 3. $(10^2)^{-3} = 10^{(2\times(-3))} = 10^{-6}$, which is $1/1,000,000$.

$\square$

Problem 1.
Beginning with the innermost bracket, we have, using rule 3,

$$(a^{-3})^{\frac{2}{3}} = a^{-3\times\frac{2}{3}} = a^{-2}.$$

Then

$$\left[\left(a^{-3}\right)^{\frac{2}{3}}\right]^{\frac{1}{2}} = \left[a^{-2}\right]^{\frac{1}{2}} = a^{-2\times\frac{1}{2}} = a^{-1}$$

$\square$

Problem 2.

Beginning again with the innermost bracket, and using rule 3, we have

$$(x^{-\frac{1}{4}})^8 = x^{-\frac{1}{4}\times 8} = x^{-2}\,.$$

Now if we use rule 3 again we have

$$[x^{-2}]^{\frac{2}{3}} = x^{-2\times\frac{2}{3}} = x^{-\frac{4}{3}}\,.$$

$\square$

Problem 3.
We have

$$\left(x^{\frac{1}{2}}\right)^3 \times \left(x^{-\frac{1}{3}}\right)^2 = x^{\frac{3}{2}} \times x^{-\frac{2}{3}}$$

using rule 3. Now we may use rule 1 and

$$x^{\frac{3}{2}} \times x^{-\frac{2}{3}} = x^{\frac{3}{2}-\frac{2}{3}} = x^{\frac{5}{6}}$$

$\square$

Problem 4.
Starting with the first term

$$\sqrt[4]{x^3} = \left(x^3\right)^{\frac{1}{4}} = x^{\frac{3}{4}}\,.$$

Thus

$$\left(\sqrt[4]{x^3}\right)^{\frac{2}{3}} = \left(x^{\frac{3}{4}}\right)^{\frac{2}{3}} = x^{\frac{3}{4}\times\frac{2}{3}} = x^{\frac{2}{4}} = x^{\frac{1}{2}}$$

Similarly,

$$\sqrt[5]{x^6} = \left(x^6\right)^{\frac{1}{5}} = x^{6\times\frac{1}{5}} = x^{\frac{6}{5}}$$

so that

$$\left(\sqrt[5]{x^6}\right)^{\frac{5}{12}} = \left(x^{\frac{6}{5}}\right)^{\frac{5}{12}} = x^{\frac{6}{5}\times\frac{5}{12}} = x^{\frac{1}{2}}\,.$$

Now we have

$$\left(\sqrt[4]{x^3}\right)^{\frac{2}{3}} \times \left(\sqrt[5]{x^6}\right)^{\frac{5}{12}} = x^{\frac{1}{2}} \times x^{\frac{1}{2}} = x^1 = x,\ .$$

$\square$

Problem 5.

The first term simplifies as follows.

$$\left(\frac{a^2}{b^3}\right)^{\frac{1}{3}} = \frac{\left(a^2\right)^{\frac{1}{3}}}{\left(b^3\right)^{\frac{1}{3}}} = \frac{a^{\frac{2}{3}}}{b} = a^{\frac{2}{3}} b^{-1} \,.$$

Treating the second term,

$$\left(\frac{b^2}{a^3}\right)^{\frac{1}{2}} = \frac{\left(b^2\right)^{\frac{1}{2}}}{\left(a^3\right)^{\frac{1}{2}}} = \frac{b}{a^{\frac{3}{2}}} = ba^{-\frac{3}{2}} \,.$$

Thus

$$\left(\frac{a^2}{b^3}\right)^{\frac{1}{3}} \times \left(\frac{b^2}{a^3}\right)^{\frac{1}{2}} = a^{\frac{2}{3}} b^{-1} \times ba^{-\frac{3}{2}} = a^{\frac{2}{3}-\frac{3}{2}} = a^{-\frac{5}{6}}.$$

$\square$